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Infinite Dimensional Control Problems I: On the Closure of the Set of Attainable States for Linear Systems

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I. INTRODUCTION

We shall be interested in studying optimal control problems in which the state space is infinite dimensional. Examples of such problems are provided by systems with "distributed parameters," systems with time delays, and certain classes of stochastic systems including several types of diffusion processes. We shall attempt to develop results analogous to those for finite dimensional systems (see, for example, [1-4]).

As an introduction to this study, we shall consider, in this paper, the problem of transferring a linear system from one state to another in minimum time with a limitation on available control power. It has been shown that the time-optimal control is (generally) "bang-bang," i.e., it uses all the control power available at each instant of time. A crucial step in the proof of this result is the demonstration of the closure of the set of states which are attainable from a given state within a specified time. This will be the main result of this paper.

II. CONVENTIONS AND NOTATION

Let X and Ω be real Banach spaces and let $E = [t_0, t_1]$ be a given closed interval on the real line. We shall use the theory of integration of Banach space valued functions as developed in [5, p. 101 ff.]. *All integrals on E will be taken with respect to Lebesgue measure* and we shall denote the integral of an X or Ω valued function $f(\tau)$ on E by $\int_{t_0}^{t_1} f(\tau) d\tau$. If v is an element of a Banach

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space, then we denote the norm of v by $\|v\|$. We denote by $L_p(E, \Omega)$ the space of equivalence classes¹ of Lebesgue measurable functions f on E to Ω such that $\|f(\tau)\|^p$ is integrable; and, in a similar manner, we denote by $L_p(E, X)$, $p \geq 1$ the space of equivalence classes of Lebesgue measurable functions f on E to X such that $\|f(\tau)\|^p$ is integrable. If f is an element of either $L_p(E, \Omega)$ or $L_p(E, X)$, then we use $\|f\|_p$ to denote the norm of f .

We let $\mathcal{L}(\Omega, X)$ denote the space of continuous linear transformations of Ω into X and, similarly, we let $\mathcal{L}(X, X)$ denote the space of continuous linear transformations of X into itself.

If A is a subset of a Banach space, then we let $co(A)$ denote the convex hull of A and we let \bar{A} denote the *strong* closure of A . Finally, we say that a bounded function f from E into a Banach space is *regulated* if it has one-sided limits at every point [6, p. 139] or equivalently, if it is the limit of a uniformly convergent sequence of step functions.

III. THE FIRST BASIC LEMMA

Let U be a closed, bounded, convex subset of Ω and let $\mathcal{U} = \{u: u \text{ is a measurable function on } E \text{ to } \Omega \text{ with } u(\tau) \in U \text{ for all } \tau \in E\}$. We call the elements of \mathcal{U} *admissible controls*. Let $K(\tau, \omega)$ be a mapping of $E \times \Omega$ into X such that the following conditions are satisfied:

- (a) For each fixed τ in E , the mapping K_τ of Ω into X given by $K_\tau(\omega) = K(\tau, \omega)$ is linear.
- (b) There is an $M > 0$ such that $\|K(\tau, \omega)\| \leq M \|\omega\|$.
- (c) For each fixed ω in Ω , the mapping K_ω of E into X given by $K_\omega(\tau) = K(\tau, \omega)$ is regulated.
- (d) If $f(\tau)$ is a measurable function on E to Ω , then $K(\tau, f(\tau))$ is a measurable mapping of E into X .

We note that if f is an element of $L_1(E, \Omega)$, then $K(\tau, f(\tau))$ is an element of $L_1(E, X)$ by virtue of conditions (c) and (d) and [5, Theorem 22, p. 117]. If \mathcal{W} is a subset of $L_1(E, \Omega)$, then we let $\mathcal{A}(E, \mathcal{W})$ be the subset of X defined by

$$\mathcal{A}(E, \mathcal{W}) = \left\{ \int_{t_0}^{t_1} K(\tau, w(\tau)) d\tau : w \in \mathcal{W} \right\} \quad (1)$$

We observe that $\mathcal{A}(E, \mathcal{U})$ is defined since every admissible control u is in

¹ Two measurable functions are equivalent if they differ by a null function. We shall usually slur over the distinction between a function f and its equivalence class.

$L_1(E, \Omega)$ as the boundedness of U implies that $\|u(\tau)\|$ is uniformly bounded on E .

We assume from now on that the space $L_2(E, \Omega)$ is reflexive.²

LEMMA 1. $\mathcal{A}(E, \mathcal{U})$ is closed in X .

PROOF. We first note that \mathcal{U} is a bounded subset of $L_2(E, \Omega)$. For, let $N > 0$ be such that U is contained in the sphere of radius N about the origin in Ω . Then, u an element of \mathcal{U} implies that $\|u(\tau)\|^2 \leq N^2$ for all τ in E and hence that $\|u(\tau)\|^2$ is integrable with

$$\left(\int_{t_0}^{t_1} \|u(\tau)\|^2 d\tau \right)^{1/2} \leq N \sqrt{t_1 - t_0} \quad (2)$$

We next observe that \mathcal{U} is a *closed* subset of $L_2(E, \Omega)$. For, suppose that u_n is a sequence in \mathcal{U} which converges to an element v in $L_2(E, \Omega)$. Then u_n converges to v in measure [5, Theorem 6, p. 122] and it follows that a subsequence u_{n_k} converges to v almost everywhere [5, corollary 13 and Theorem 15, p. 150]. Since U is closed in Ω , $v(\tau)$ is an element of U for almost all τ in E . In other words, v differs from an element of \mathcal{U} by a null function, and so v is an element of \mathcal{U} (where we recall that the elements of $L_2(E, \Omega)$ are, strictly speaking, equivalence classes).

Since U is convex, it follows from condition (a) on K that \mathcal{U} is convex. Therefore \mathcal{U} is weakly-closed in $L_2(E, \Omega)$ [5, Theorem 13, p. 422] and hence is *weakly-compact* in $L_2(E, \Omega)$ as $L_2(E, \Omega)$ is reflexive [5, Corollary 8, p. 425]. In view of the Hölder inequality [5, Lemma 2, p. 119] and the fact that the measure of E is finite, we can assert that $L_2(E, \Omega)$ is contained in $L_1(E, \Omega)$. Now we let ψ be the mapping of $L_1(E, \Omega)$ into X defined by:

$$\psi(f) = \int_{t_0}^{t_1} K(\tau, f(\tau)) d\tau, \quad f \in L_1(E, \Omega) \quad (3)$$

ψ is a linear mapping of $L_1(E, \Omega)$ into X by virtue of condition (a) on K . We will show that the restriction of ψ to $L_2(E, \Omega)$ is continuous and, therefore, *weakly-continuous* [5, Theorem 15, p. 422]. The lemma will follow immediately since $\mathcal{A}(E, \mathcal{U}) = \psi(\mathcal{U})$ will then be convex and weakly-closed (being weakly-compact) and, therefore, closed [5, Theorem 13, p. 422].

² This assumption is satisfied, for example, when Ω is a separable, reflexive space [7, p. 134], or a Hilbert space, or a finite dimensional space.

Now if f is an element of $L_1(E, \Omega)$, then

$$\|\psi(f)\| \leq \int_{t_0}^{t_1} \|K(\tau, f(\tau))\| d\tau \quad (4)$$

which implies that (condition (b) on K)

$$\|\psi(f)\| \leq M \int_{t_0}^{t_1} \|f(\tau)\| d\tau = M \|f\|_1 \quad (5)$$

It follows from the Hölder inequality and the inequality (5) that if g is an element of $L_2(E, \Omega)$, then

$$\|\psi(g)\| \leq M \sqrt{t_1 - t_0} \|g\|_2 \quad (6)$$

Therefore, the restriction of ψ to $L_2(E, \Omega)$ is continuous and the proof of the lemma is complete.

COROLLARY 1. *If U is a sphere in Ω , then $\mathcal{A}(E, \mathcal{U})$ is closed.*

COROLLARY 2. *If U is replaced by a family $\{U_\tau : \tau \in E\}$ of closed, convex sets in Ω which are all contained in a given sphere in Ω , then $\mathcal{A}(E, \mathcal{U})$ is closed (where \mathcal{U} is now the set $\{u : u \text{ a measurable function on } E \text{ to } \Omega \text{ with } u(\tau) \in U_\tau \text{ for all } \tau \text{ in } E\}$).*

COROLLARY 3. *If \mathcal{W} is weakly-compact in $L_2(E, \Omega)$ (or in $L_1(E, \Omega)$), then $\mathcal{A}(E, \mathcal{W})$ is weakly-closed in X .*

IV. THE SECOND BASIC LEMMA

We have shown in the course of the proof of Lemma 1 that the set \mathcal{U} of admissible controls is closed, bounded, convex, and weakly-compact in $L_2(E, \Omega)$ (under the assumption that $L_2(E, \Omega)$ is reflexive). Let \mathcal{W} be a closed, bounded, weakly-compact subset of $L_2(E, \Omega)$ and let $e(\mathcal{W})$ denote the set of extremal points of \mathcal{W} [5, p. 439].

LEMMA 2. *If \mathcal{W}_1 is a subset of $L_2(E, \Omega)$ such that*

$$e(\mathcal{W}) \subset \mathcal{W}_1 \subset \mathcal{W} \quad (7)$$

then

$$\mathcal{A}(E, \overline{\text{co}}(\mathcal{W}_1)) = \overline{\text{co}}(\mathcal{A}(E, \mathcal{W})),$$

PROOF. Since \mathcal{W} is weakly-compact, it follows from a theorem of Krein-Smulian [5, Theorem 4, p. 434] that $\overline{\text{co}(\mathcal{W})}$ is weakly-compact and hence that $e(\overline{\text{co}(\mathcal{W})}) = e(\mathcal{W})$ [5, Lemma 5, p. 440]. It then follows from a theorem of Krein-Milman [5, theorem 4, p. 440] that the *weak*-closure of $\text{co}(e(\mathcal{W}))$ is the same as the weak closure of $\overline{\text{co}(\mathcal{W})}$ as $\overline{\text{co}(\mathcal{W})}$ is convex. In view of the fact that for convex subsets of a locally convex topological vector space, weak and strong closure are equivalent [5, Theorem 13, p. 422], we see that

$$\overline{\text{co}(e(\mathcal{W}))} = \overline{\text{co}(\mathcal{W})} \quad (8)$$

and hence, that

$$\overline{\text{co}(\mathcal{W}_1)} = \overline{\text{co}(\mathcal{W})} \quad (9)$$

as

$$\text{co}(e(\mathcal{W})) \subset \text{co}(\mathcal{W}_1) \subset \text{co}(\mathcal{W}).$$

We now claim that

$$\mathcal{A}(E, \overline{\text{co}(\mathcal{W}_1)}) = \overline{\mathcal{A}(E, \text{co}(\mathcal{W}_1))} \quad (10)$$

In view of the proof of Lemma 1, we see that $\mathcal{A}(E, \overline{\text{co}(\mathcal{W}_1)})$ is closed, and therefore that $\mathcal{A}(E, \overline{\text{co}(\mathcal{W}_1)})$ contains $\overline{\mathcal{A}(E, \text{co}(\mathcal{W}_1))}$. On the other hand, if x is an element of $\mathcal{A}(E, \overline{\text{co}(\mathcal{W}_1)})$, then there is an f in $\overline{\text{co}(\mathcal{W}_1)}$ with $x = \psi(f)$. If f_n is a sequence of elements of $\text{co}(\mathcal{W}_1)$ which converges to f and if $x_n = \psi(f_n)$, then x_n is a sequence of elements of $\mathcal{A}(E, \text{co}(\mathcal{W}_1))$ which converges to x since ψ is *continuous* (Eq. (6)). Hence x is in $\overline{\mathcal{A}(E, \text{co}(\mathcal{W}_1))}$ and (10) is established.

However, it is clear from condition (a) on K that

$$\mathcal{A}(E, \text{co}(\mathcal{W}_1)) = \text{co}(\mathcal{A}(E, \mathcal{W}_1)) \quad (11)$$

The lemma is an immediate consequence of the relations (9), (10), and (11).

COROLLARY 4. *If $\mathcal{A}(E, \mathcal{W}_1)$ is convex and closed, then*

$$\mathcal{A}(E, \mathcal{W}_1) = \mathcal{A}(E, \mathcal{W})$$

and hence $\mathcal{A}(E, \mathcal{W})$ is closed.

PROOF. If $\mathcal{A}(E, \mathcal{W}_1)$ is convex and closed, then

$$\begin{aligned} \mathcal{A}(E, \mathcal{W}_1) &= \overline{\mathcal{A}(E, \mathcal{W}_1)} = \overline{\text{co}(\mathcal{A}(E, \mathcal{W}_1))} = \overline{\mathcal{A}(E, \text{co}(\mathcal{W}_1))} \\ &= \mathcal{A}(E, \overline{\text{co}(\mathcal{W}_1)}) = \overline{\text{co}(\mathcal{A}(E, \mathcal{W}))}. \end{aligned}$$

Therefore, $\mathcal{A}(E, \mathcal{W}_1)$ contains $\mathcal{A}(E, \mathcal{W})$ and since $\mathcal{A}(E, \mathcal{W})$ contains $\mathcal{A}(E, \mathcal{W}_1)$, the corollary is proved.

COROLLARY 5. *If X and Ω are finite dimensional, if U is a sphere or a cube and if $\hat{\mathcal{U}} = \{u: u \in \mathcal{U} \text{ and } u(\tau) \in e(U) \text{ for all } \tau \text{ in } E\}$, then $\mathcal{A}(E, \hat{\mathcal{U}}) = \mathcal{A}(E, \mathcal{U})$.*

PROOF. Since it is well-known that $\mathcal{A}(E, \hat{\mathcal{U}})$ is closed and convex (see, for example, [8, 3, 4]), it will suffice to show that $e(\mathcal{U})$ is contained in $\hat{\mathcal{U}}$. Suppose that u is an element of \mathcal{U} which is not equivalent to an element of $\hat{\mathcal{U}}$, then there is a set of positive measure E_1 contained in E for which $u(\tau) \notin e(U)$. It follows that there is an $\omega \neq 0$ in Ω such that $u(\tau) + \omega$ and $u(\tau) - \omega$ are in U for τ in a set of positive measure E_2 contained in E_1 . If h_1 and h_2 are defined by

$$h_1(\tau) = \begin{cases} u(\tau) & \text{on } E - E_2 \\ u(\tau) + \omega & \text{on } E_2 \end{cases} \quad (12a)$$

$$h_2(\tau) = \begin{cases} u(\tau) & \text{on } E - E_2 \\ u(\tau) - \omega & \text{on } E_2 \end{cases} \quad (12b)$$

then h_1 and h_2 are in \mathcal{U} , are not both equivalent to u ; and have the property that

$$\frac{1}{2} h_1 + \frac{1}{2} h_2 = u \quad (13)$$

Thus, u is not an element of $e(\mathcal{U})$ which shows that the complement of $\hat{\mathcal{U}}$ in \mathcal{U} is contained in the complement of $e(\mathcal{U})$ in \mathcal{U} and the corollary is established.

We observe that Corollary 5 is the basis for the ‘‘bang-bang’’ principle (see, for example, [3]). We shall derive a result which is analogous to this corollary in the next section.

V. A WEAK-DENSITY LEMMA

Suppose that U is a closed, bounded, convex, weakly-compact subset of Ω and let E and \mathcal{U} be as in Section III. Let $\hat{\mathcal{U}}$ be the subset of \mathcal{U} defined by

$$\hat{\mathcal{U}} = \{u \in \mathcal{U}: u(\tau) \in e(U) \text{ for all } \tau \text{ in } E\} \quad (14)$$

Then we have:

LEMMA 3. *The weak-closure of $\mathcal{A}(E, \hat{\mathcal{U}})$ is $\mathcal{A}(E, \mathcal{U})$.*

PROOF. Let $s(\mathcal{U})$ denote the set of simple functions in \mathcal{U} . If $x = \psi(u)$ with u in \mathcal{U} , then there is a sequence u_n in $s(\mathcal{U})$ such that u_n converges weakly to u (since $s(\mathcal{U})$ is strongly dense in \mathcal{U}). Since ψ is weakly-continuous,

it follows that $\psi(u_n)$ converges weakly to $\psi(u) = x$. We will show that if $y \in \mathcal{A}(E, s(\mathcal{U}))$, then there is a sequence y_n in $\mathcal{A}(E, \mathcal{U})$ which converges weakly to y . It will follow that the weak-closure of $\mathcal{A}(E, \mathcal{U})$ contains the weak-closure of $\mathcal{A}(E, s(\mathcal{U}))$ which contains $\mathcal{A}(E, \mathcal{U})$. The lemma will then be an immediate consequence of the fact that $\mathcal{A}(E, \mathcal{U})$ is weakly-closed.

If $y \in \mathcal{A}(E, s(\mathcal{U}))$, then there is a simple function f in $s(\mathcal{U})$ with $y = \psi(f)$. Suppose that

$$f = \sum_{i=1}^n \omega_i \chi_i \quad (15)$$

where χ_i is the characteristic function of a measurable set E_i contained in E , $E_i \cap E_j$ is empty for $i \neq j$, $\bigcup_{i=1}^n E_i = E$, and the ω_i are elements of U . We shall show that there is a sequence h_m^i in $L_2(E, \Omega)$ such that $h_m^i(\tau) = 0$ if $\tau \notin E_i$ and $h_m^i(\tau) \in e(U)$ if $\tau \in E_i$ and such that h_m^i converges weakly to the function $\omega_i \chi_i$. Then the sequence $h_m = \sum_{i=1}^n h_m^i$ will converge weakly to f and $h_m(\tau) = h_m^i(\tau)$ for $\tau \in E_i$ will imply that $h_m(\tau) \in e(U)$ for all τ in E (i.e., that $h_m \in \mathcal{U}$).

Consider, for example, $\omega_1 \chi_1$. Since U is weakly-compact and convex, it follows that there is a sequence λ_k^1 in $\text{co}(e(U))$ which converges strongly to ω_1 [5, Theorem 4, p. 440 and Theorem 13, p. 422]. We have

$$\lambda_k^1 = \sum_{j=1}^{v_k} a_{kj}^1 \omega_{kj}^1 \quad (16)$$

with

$$a_{kj}^1 \geq 0, \quad \sum_{j=1}^{v_k} a_{kj}^1 = 1, \quad \text{and} \quad \omega_{kj}^1 \in e(U).$$

We set

$$h_{kj}^1 = \omega_{kj}^1 \chi_1 \quad \text{and} \quad g_k^1 = \sum_{j=1}^{v_k} a_{kj}^1 h_{kj}^1.$$

Then it follows that g_k^1 converges strongly to $\omega_1 \chi_1$. Let h_m^1 be the sequence

$$h_1^1 = h_{11}^1, h_2^1 = h_{12}^1, \dots, h_{v_1}^1 = h_{1v_1}^1, h_{v_1+1}^1 = h_{21}^1, \dots, \text{etc.} \quad (17)$$

We claim that h_m^1 converges weakly to $\omega_1 \chi_1$. It will be sufficient to show that $\omega_1 \chi_1$ is an element of $\text{co}\{h_m^1, h_{m+1}^1, \dots\}$ for every m [5, Exercise 43, p. 439]. However, for every m , there is a $k_0(m)$ such that $k \geq k_0(m)$ implies that g_k^1 is an element of $\text{co}\{h_m^1, \dots\}$ from which our claim follows. We note that

$h_m^1(\tau) = 0$, if $\tau \notin E_1$ and that $h_m^1(\tau) \in e(U)$ for τ in E_1 . We may construct the sequences h_m^i for $i = 2, \dots, n$ in a similar manner and the lemma is established.

In view of this lemma, let us look at the following rather interesting example.

Example 1. Let Ω be the space of real numbers with the norm of an element ω of Ω given by $\|\omega\| = \sqrt{\omega^2}$. Let $U = \{\omega \in \Omega : \|\omega\| \leq 1\}$ and let X be the space $L_2([0, 2\pi], \Omega)$. Let ϕ_n be an orthonormal system on $[0, 2\pi]$, $n = 0, \pm 1, \dots$ and let

$$E_n(s, t) = \sum_{-n}^n \phi_n(s) \phi_n(t)$$

and suppose that the ϕ_n are uniformly bounded. Let E be the interval $[0, 2\pi]$ and define the mapping $K_n(\tau, \omega)$ of $E \times R$ into X as follows:

$$K_n(\tau, \omega)(\sigma) = E_n(\sigma, \tau) \omega \quad (18)$$

It is easy to see that $K_n(\tau, \omega)$ satisfies the conditions (a), (b), (c), and (d) of Section III. If we let ψ_n be the mapping of X into itself defined by

$$\psi_n(f)(s) = \int_0^{2\pi} K_n(\tau, f(\tau))(s) d\tau, \quad f \in L_2([0, 2\pi], \Omega) \quad (19)$$

then it follows that ψ_n converges to the identity I in $\mathcal{L}(X, X)$ [5, Exercise 3, p. 358]. Let ϵ be a small positive number and suppose that $\|\psi_{n_0} - I\| < \epsilon$. We set $K(\tau, \omega) = K_{n_0}(\tau, \omega)$ and $\psi = \psi_{n_0}$. We now claim that

$$\mathcal{A}(E, \hat{\mathcal{U}}) \neq \mathcal{A}(E, \mathcal{U}).$$

Let $u(\tau) = \frac{1}{2}$ for all τ in E . Then $u \in \mathcal{U}$ and we claim that $\|u - \hat{u}\|_2 > 0$ for all \hat{u} in $\hat{\mathcal{U}}$ for we have, for any \hat{u} ,

$$\|u - \hat{u}\|_2^2 = \int_0^{2\pi} |u(s) - \hat{u}(s)|^2 ds \quad (20a)$$

$$\geq \int_0^{2\pi} ||u(s)| - |\hat{u}(s)||^2 ds \quad (20b)$$

$$\geq \int_0^{2\pi} \left| \frac{1}{2} - 1 \right|^2 ds = \frac{\pi}{2} \quad (20c)$$

This implies that $\psi(u) \neq \psi(\hat{u})$ for any \hat{u} in \mathcal{U} since: if there were a \hat{u}_0 such that $\psi(u) = \psi(\hat{u}_0)$, then $\|(\psi - I)(u - \hat{u}_0)\|_2 = \|u - \hat{u}_0\|_2$ would be no greater than $\|\psi - I\| \cdot \|u - \hat{u}_0\|_2$ which is no greater than $\epsilon \cdot \|u - \hat{u}_0\|_2$.

This example shows that the "bang-bang" principle [3, 4] is *not valid* for infinite dimensional systems even in the case where Ω and X are separable Hilbert spaces.

VI. LINEAR SYSTEMS

Suppose that we are given a controlled system whose behavior is described by a trajectory $x(t, v)$ in X of the form

$$x(t, v) = \Phi(t) x_0 + \Phi(t) \int_{t_0}^{t_1} K(\tau, v(\tau)) d\tau + \Phi(t) \int_{t_0}^{t_1} \Phi^{-1}(\tau) f(\tau) d\tau \quad (21)$$

where the following conditions are satisfied:

(α) $\Phi(s)$, $s \geq t_0$ is a linear homeomorphism of X into itself such that the mapping $s \rightarrow \Phi(s)$ of $[t_0, \infty)$ into $\mathcal{L}(X, X)$ is regulated and bounded on every bounded subinterval of $[t_0, \infty)$.

(β) K satisfies the conditions (a), (b), (c), and (d) of Section III on every subinterval $[t_0, t_1]$ of $[t_0, \infty)$.

(γ) $f(\tau)$ is a regulated function on $[t_0, \infty)$ to X which is bounded on every bounded subinterval of $[t_0, \infty)$.

(δ) $v(\tau)$ is a measurable mapping of every subinterval $[t_0, t_1]$ of $[t_0, \infty)$ into Ω such that $v(\tau) \in V$ for all τ where V is a closed, bounded, convex, weakly-compact subset of Ω containing the origin of Ω .

A function v which satisfies condition (δ) will be called an *admissible control* and we let \mathcal{V} denote the set of admissible controls. Let $\xi(\tau)$ be a regulated function on $[t_0, \infty)$ to X with $\xi(t_0) \neq x_0$.

PROPOSITION 1 [3]. *If there is a $t_1 > t_0$ such that $x(t_1, v) = \xi(t_1)$ for some v in \mathcal{V} , then there is a v^* in \mathcal{V} such that*

$$x(t^*, v^*) = \xi(t^*) \quad \text{for some} \quad t^* > t_0 \quad (22)$$

and

$$x(t, v) \neq \xi(t) \quad (23)$$

for $t_0 < t < t^*$ and all v in \mathcal{V} .

The proof is essentially the same as that given for a similar result in [3, p. 8] and is, therefore, omitted.

COROLLARY 6. *If $\xi(\tau) \equiv 0$, and if $f(\tau) \equiv 0$, then the time optimal control problem for the system (21) has a solution for a given initial state $x_0 \neq 0$ if and only if there is a $t_1 > t_0$ such that $-x_0$ is an element of $\mathcal{A}([t_0, t_1], \mathcal{V})$.*

Therefore, we see that the question of whether or not the time optimal control problem for systems of the form (21) has a solution is intimately related to the problem of determining the set

$$\mathcal{A}(\mathcal{V}) = \bigcup_{t > t_0} \mathcal{A}([t_0, t], \mathcal{V}).$$

This problem will be examined in a later paper on controllability.

COROLLARY 7 [2]. *If $\xi(\tau)$ and $f(\tau)$ are identically zero, if $\Omega = X$, if the origin is an interior point of V , if $\Phi(t) = e^{tA}$ where A is an invertible element of $\mathcal{L}(X, X)$ which has the property that*

$$\lim_{t \rightarrow \infty} \|e^{tA}\| = 0 \quad (24)$$

and if $K(\tau, v(\tau)) = e^{-\tau A} v(\tau)$, then the time optimal control problem has a solution.

PROOF. We suppose for simplicity that $t_0 = 0$ and we let $x_0 \neq 0$ be a given initial state. Let $t^* > 0$ be such that

$$\|\omega\| \leq \frac{\|e^{t^* A}\|}{1 - \|e^{t^* A}\|} \|A\| \cdot \|x_0\| \quad (25)$$

implies that ω is in V . Such a t^* exists in view of (24) and the assumption that the origin is an interior point of V . We set

$$\omega^* = \sum_{n=0}^{\infty} (e^{t^* A})^{n+1} A x_0 \quad (26a)$$

$$= e^{t^* A} (I - e^{t^* A})^{-1} A x_0 \quad (26b)$$

Then ω^* is in V and the function $v^*(\tau) = \omega^*$ for τ in $[0, t^*]$ is in \mathcal{V} . We claim that

$$-x_0 = \int_0^{t^*} e^{-\tau A} v^*(\tau) d\tau \quad (27)$$

To see this, we observe that

$$\begin{aligned} \int_0^{t^*} e^{-\tau A} v^*(\tau) d\tau &= \int_0^{t^*} e^{-\tau A} \omega^* d\tau \\ &= t^* \omega^* - \frac{t^{*2}}{2!} A \omega^* + \dots \end{aligned} \quad (28b)$$

$$= A^{-1}(I - e^{-t^* A}) \omega^* \quad (28c)$$

$$= A^{-1}(e^{t^* A} - I)(I - e^{t^* A})^{-1} A x_0 \quad (28d)$$

$$= -x_0 \quad (28e)$$

The corollary then follows from Corollary 6.

REFERENCES

1. KALMAN, R. E., HO, Y. C. AND NARENDRA, K. S. Controllability of linear dynamical systems. *Contrib. Differential Eqs.* 1 (1962).
2. BELLMAN, R., GLICKSBERG, I. AND GROSS, O. On the "bang-bang" control problem. *Quart. Appl. Math.* 14 (1956).
3. LASALLE, J. P. The time optimal control problem. "Contributions to the Theory of Nonlinear Oscillations," Vol. V. Princeton Univ. Press, Princeton, N.J., 1960.
4. NEUSTADT, L. W. The existence of optimal controls in the absence of convexity conditions. *J. Math. Anal. Appl.* 7 (1963).
5. DUNFORD, N. AND SCHWARTZ, J. T. "Linear Operators, Part I: General Theory." Interscience, New York, 1958.
6. DIEUDONNE, J. "Foundations of Modern Analysis." Academic Press, New York, 1960.
7. DIEUDONNE, J. Sur le theoreme de Lebesgue-Nikodym V. *Can. J. Math.* 3 (1951).
8. BLACKWELL, D. The range of certain vector integrals. *Proc. Am. Math. Soc.* 2 (1951).